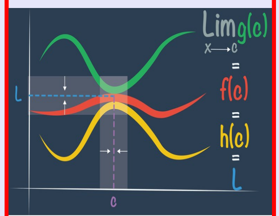
Calculus I

Lecture 9



Feb 19-8:47 AM

First derivative f', y', $\frac{dy}{dx}$

Second derivative f'', y'', $\frac{d^2y}{dx^2}$

First derivative tells us where graph is $\boxed{\text{increasing}}$ or $\boxed{\text{decreasing}}$.

$f'>0$ (increasing) $f'<0$ (decreasing)

Second derivative tells us where graph is $\boxed{\text{Concave up}}$ $\cup$ or

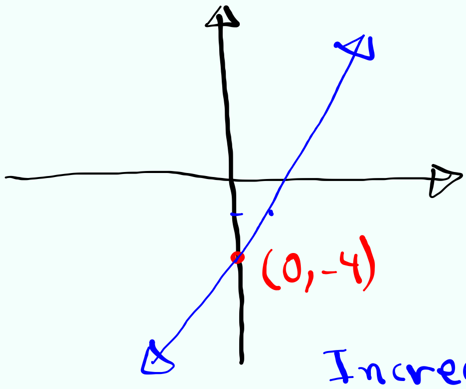
$f''>0$

$\boxed{\text{Concave down}}$ $\cap$

$f''<0$

Jan 20-8:28 AM

$f(x) = 2x - 4$



Increasing

$f'(x) = 2 > 0$

$f(x)$ increasing

$f''(x) = \frac{d}{dx}[f'(x)]$

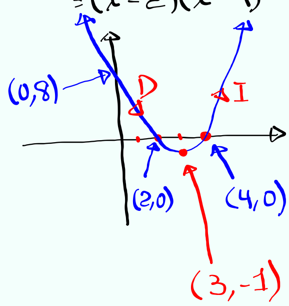
$= 0$

$f(x)$ has no Concavity

Jan 20-8:33 AM

$f(x) = x^2 - 6x + 8$

$= (x-2)(x-4)$



$f'(x) = 2x - 6$

$f'(x) = 0$ or undefined gives us Critical Points

$2x - 6 = 0$

$x = 3$ C.N.

$f(3) = 3^2 - 6(3) + 8$

$= -1$

$(3,-1)$ C.P.

$f''(x) = 2 > 0$

Concave up

Sign Chart

$x=2$ $\qquad$ $x=4$

x	$-\infty$	3	∞
$f'(x)$	$-$	•	$+$
$f''(x)$	$+$		$+$
$f(x)$			

Jan 20-8:35 AM

$f(x) = x^3 - 27x + 1$

Polynomial function $\to$ Cont. $(-\infty, \infty)$

$f'(x) = 3x^2 - 27$

$f'(x) = 0$

$3x^2 - 27 = 0$

$f''(x) = 6x$

$3(x^2 - 9) = 0$

$f''(x) = 0$ or undefined gives us Possible inflection Pts. (Concavity changes)

$\boxed{3(x+3)(x-3)} = 0$

$x = -3$, $x = 3$ C.N.

$(-3, f(-3))$, $(3, f(3))$ C.P.

$(-3, +)$, $(3, -)$

$f''(x) = 0$

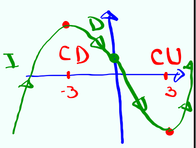
$6x = 0$ $\quad x = 0$

P.I.P. $(0, f(0)) = (0, 1)$

Sign chart

$x=-4$ $\quad x=0$ $\quad x=4$

x	$-\infty$	-3	0	3	∞
$f'(x)$	+	–	–	+	
$f''(x)$	–	–	+	+	
$f(x)$	↗	↘	↘	↗	



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$f(x) = x^4 - 2x^2$

1) Domain $(-\infty, \infty)$

2) Y-Int $(0,0)$

3) x-Ints $(\pm\sqrt{2}, 0), (0,0)$

$x^4 - 2x^2 = 0$

$x^2(x^2 - 2) = 0$

4) Continuity $(-\infty, \infty)$

5) $f'(x) = 4x^3 - 4x$

$f'(x) = 0$

$4x^3 - 4x = 0$

$4x(x^2 - 1) = 0$

C.N. $0, \pm 1$

C.P. $(0,0), (1,-1), (-1,-1)$

6) $f''(x) = 12x^2 - 4$

$f''(x) = 0$

$12x^2 - 4 = 0$

$x^2 = \frac{4}{12} = \frac{1}{3}$

$x = \pm\frac{1}{\sqrt{3}}$

P.I.P.

$\left(\pm\frac{1}{\sqrt{3}}, \quad\right)$

Sign Chart

x	$-\infty$	-1	$\frac{-1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	∞
$f'(x)$	–	+	+	–	–	+	
$f''(x)$	+	+	–	–	+	+	
$f(x)$	↘	↗	↗	↘	↘	↗	

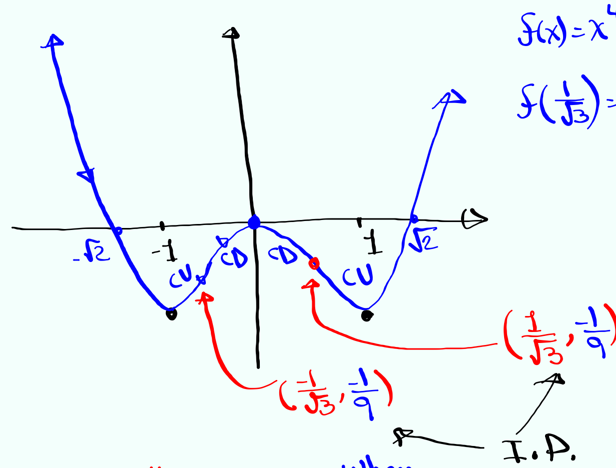
Jan 20-8:59 AM

$f(x)=x^4-2x^2$

$f\left(\frac{1}{\sqrt{3}}\right)=\left(\frac{1}{\sqrt{3}}\right)^4-2\left(\frac{1}{\sqrt{3}}\right)^2$

$=\frac{1}{9}-\frac{2}{9}$

$=\frac{-1}{9}$



$f(-x)=(-x)^4-2(-x)^2$

$=x^4-2x^2$

$=f(x)$

when $f(-x)=f(x)$

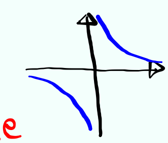
then

$f(x)$ is an even function

makes it symmetric with respect to y-axi

Jan 20-9:14 AM

$f(x)=\frac{1}{x}$



1) Domain $x\neq 0$ $(-\infty,0)\cup(0,\infty)$

2) y-Int None

3) x-Int $\frac{1}{x}=0$ None

4) Continuity $(-\infty,0)\cup(0,\infty)$

5) $f'(x)=\frac{-1}{x^2}<0$

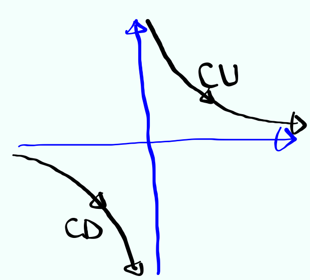
$f'(x)=0$ or undefined at $x=0$

6) $f''(x)=\frac{2}{x^3}$

$f''(x)=0$ or undefined at $x=0$

Sign Chart

x	$-\infty$	0	∞
$f'(x)$	$-$	0	$-$
$f''(x)$	$-$	0	$+$
$f(x)$	↘		↘



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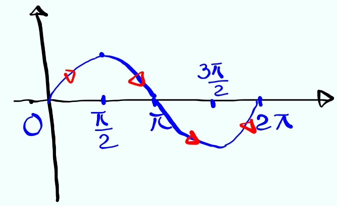
$f(x) = \sin x \quad , \quad [0, 2\pi]$



$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'(x) = 0 \quad \cos x = 0 \quad x = \frac{\pi}{2} \; , \; x = \frac{3\pi}{2}$

$f''(x) = 0 \quad -\sin x = 0 \quad x = 0 \; , \; x = \pi \, , \; x = 2\pi$

Sign chart

x	0		$\frac{\pi}{2}$		π		$\frac{3\pi}{2}$		2π
$f'(x)$		$+$	0	$-$		$-$	0	$+$	
$f''(x)$		$-$		$-$	0	$+$		$+$	
$f(x)$		$\nearrow$		$\searrow$		$\searrow$		$\nearrow$	

Jan 20-9:33 AM

$f(x) = \dfrac{x}{x^2 - 1}$

1) Domain

$x \neq \pm 1$

2) y-Int $(0,0)$

3) x-Int $(0,0)$

4) Continuity

everywhere except ± 1

5) $f(-x) = \dfrac{-x}{(-x)^2 - 1} = \dfrac{-x}{x^2 - 1} = -\dfrac{x}{x^2 - 1} = -f(x)$

$f(-x) = -f(x) \rightarrow f(x)$ is an odd function

symmetric w/t origin.

6) $f'(x) = \dfrac{-(x^2+1)}{(x^2-1)^2}$

7) $f''(x) = \dfrac{2x(x^2+3)}{(x^2-1)^3}$

Jan 20-9:40 AM

use chain rule to find $f'(x)$

1) $f(x) = (x - \sqrt{x})^5$

$x^{\frac{1}{2}}$

$f'(x) = 5(x - \sqrt{x})^4 \cdot \frac{d}{dx}[x - \sqrt{x}]$

$= 5(x - \sqrt{x})^4 \cdot [1 - \frac{1}{2}x^{-1/2}]$

$= 5(x - \sqrt{x})^4 \cdot (1 - \frac{1}{2\sqrt{x}})$

$= 5(x - \sqrt{x})^4 \cdot \frac{2\sqrt{x} - 1}{2\sqrt{x}}$

$= \frac{5\sqrt{x}(x - \sqrt{x})^4(2\sqrt{x} - 1)}{2x}$

Jan 20-10:25 AM

2) $f(x) = \sqrt{\sin x + \cos x}$ for $0 \leq x \leq \frac{\pi}{2}$

$= (\sin x + \cos x)^{1/2}$

$f'(x) = \frac{1}{2}(\sin x + \cos x)^{-1/2} \cdot \frac{d}{dx}[\sin x + \cos x]$

$= \frac{1}{2\sqrt{\sin x + \cos x}} \cdot (\cos x - \sin x)$

$= \boxed{\frac{\cos x - \sin x}{2\sqrt{\sin x + \cos x}}}$

Jan 20-10:30 AM

use implicit differentiation to find $\frac{dy}{dx}$.

$$\sqrt[3]{x^2} + \sqrt[3]{y^2} = 2$$

$$x^{2/3} + y^{2/3} = 2$$

$$\frac{d}{dx}\left[x^{2/3}\right] + \frac{d}{dx}\left[y^{2/3}\right] = \frac{d}{dx}[2]$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

multiply by 3, then divide by 2.

$$x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

$$y^{-1/3}\frac{dy}{dx} = -x^{-1/3}$$

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$

$$\boxed{\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}}$$

Jan 20-10:34 AM

find $\frac{dy}{dx}$ for

$$x^3y^2 - x^2y^3 + 4 = 0$$

$$\frac{d}{dx}\left[x^3y^2\right] - \frac{d}{dx}\left[x^2y^3\right] + \frac{d}{dx}[4] = \frac{d}{dx}[0]$$

$$\frac{d}{dx}[x^3]\cdot y^2 + x^3\cdot\frac{d}{dx}[y^2] - \frac{d}{dx}[x^2]y^3 - x^2\cdot\frac{d}{dx}[y^3] + 0 = 0$$

$$3x^2y^2 + x^3\cdot 2y\cdot\frac{dy}{dx} - 2xy^3 - x^2\cdot 3y^2\cdot\frac{dy}{dx} = 0$$

$$(2x^3y - 3x^2y^2)\frac{dy}{dx} = 2xy^3 - 3x^2y^2$$

$$\frac{dy}{dx} = \frac{2xy^3 - 3x^2y^2}{2x^3y - 3x^2y^2}$$

$$= \frac{xy^2(2y - 3x)}{x^2y(2x - 3y)} \quad \boxed{\frac{y(2y-3x)}{x(2x-3y)}}$$

Jan 20-10:41 AM

Use linear approximation to evaluate 2.1^5

1) by Calculator

$2.1^5 = \boxed{40.84101}$

$f(x) \approx f(a) + f'(a)(x-a)$

$f(x) = x^5$

$a = 2$

$f(2) = 2^5 = 32$

$f'(x) = 5x^4$

$f'(2) = 5(2)^4 = 5 \cdot 16 = 80$

$x^5 \approx f(2) + f'(2)(x-2)$

$x^5 \approx 32 + 80(x-2)$

$2.1^5 \approx 32 + 80(2.1-2)$

$= 32 + 80(.1)$

$= 32 + 8 \approx \boxed{40}$

Jan 20-10:51 AM

Use linear approximation to estimate $\sin 31°$

1) using Calculator $\sin 31° \approx \boxed{.515}0380749$

2) Linear approximation

$f(x) \approx f(a) + f'(a)(x-a)$

$f(x) = \sin x$

$a = 30°$

$f(30°) = \sin 30° = \frac{1}{2}$

$f'(x) = \cos x$

$f'(30°) = \cos 30° = \frac{\sqrt{3}}{2}$

$\sin x \approx f(30°) + f'(30°)(x - 30°)$

$\sin x \approx .5 + \frac{\sqrt{3}}{2}(x - 30°)$

$\sin 31° \approx .5 + \frac{\sqrt{3}}{2}(31° - 30°)$

$= .5 + \frac{\sqrt{3}}{2}(1°)$

$= .5 + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{180}$

$= .5 + \frac{\pi\sqrt{3}}{360}$

$\approx \boxed{.515}1149947$

Jan 20-10:57 AM

use linear approximation to estimate $\frac{1}{\sqrt{10}}$

by calculator $\frac{1}{\sqrt{10}} \approx \boxed{.316}227766$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$f(x) = \frac{1}{\sqrt{x}}$

$a = 9$

$f(x) = x^{-1/2}$

$f'(x) = -\frac{1}{2}x^{-3/2}$

$= \frac{-1}{2\sqrt{x^3}} = \frac{-1}{2x\sqrt{x}}$

$f(9) = \frac{1}{\sqrt{9}} = \frac{1}{3}$

$f'(9) = \frac{-1}{2\cdot 9\sqrt{9}} = \frac{-1}{54}$

$\frac{1}{\sqrt{x}} \approx f(9) + f'(9)(x-9)$

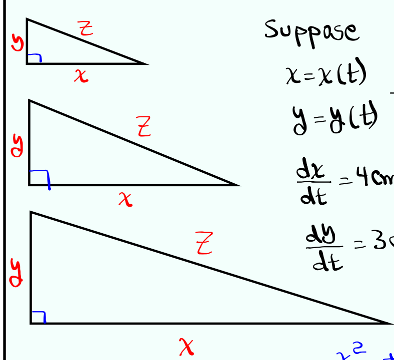
$\approx \frac{1}{3} + \frac{-1}{54}(x-9)$

$\frac{1}{\sqrt{10}} \approx \frac{1}{3} - \frac{1}{54}(10-9)$

$= \frac{1}{3} - \frac{1}{54} = \frac{18-1}{54} = \frac{17}{54}$

$\frac{17}{54} \approx \boxed{.314}8148\ldots$

Jan 20-11:07 AM



Suppose

$x = x(t)$

$y = y(t)$ $\rightarrow z = z(t)$

$\frac{dx}{dt} = 4$ cm/min.

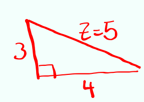
$\frac{dy}{dt} = 3$ cm/min.

find $\frac{dz}{dt}$ after 1 min.

$x^2 + y^2 = z^2$

$\frac{d}{dt}[x^2] + \frac{d}{dt}[y^2] = \frac{d}{dt}[z^2]$

$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2z\frac{dz}{dt}$



after 1 min.

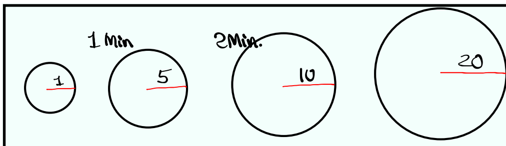
$4\cdot 4 + 3\cdot 3 = 5\frac{dz}{dt}$

$25 = 5\frac{dz}{dt}$

$\frac{dz}{dt} = 5$ cm/min

Related Rates

Jan 20-11:16 AM



Radius is increasing at 5 cm/min. $\frac{dR}{dt} = 5\text{cm/min}$

How fast is the area increasing ~~after 2 min?~~ when Radius is 10 cm.

$\frac{dA}{dt} = \quad \text{cm}^2/\text{min}$

Give me a formula between Area & Radius of any Circle. $A = \pi R^2$

$$\frac{dA}{dt} = \frac{d}{dt}\left[\pi R^2\right]$$

$$\frac{dA}{dt} = \pi \cdot 2R \cdot \frac{dR}{dt} = 2\pi R \frac{dR}{dt}$$

$$= 2\pi \cdot 10 \cdot 5$$

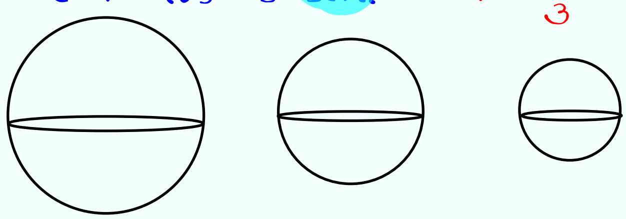
$$\boxed{\frac{dA}{dt} = 100\pi \ \text{cm}^2/\text{min.}}$$

Jan 20-11:28 AM

A basketball is losing air. $\frac{dR}{dt} = -2\,\text{cm/min}$

Its radius is getting Smaller at 2cm/min.

How fast is $\boxed{\text{its Volume}}$ changing when the radius is 5cm? $V = \frac{4\pi r^3}{3}$ sphere



$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4\pi r^3}{3}\right]$$

$$\frac{dV}{dt} = \frac{4\pi}{\cancel{3}} \cdot \cancel{3} r^2 \cdot \frac{dr}{dt}$$

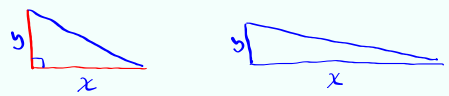
$$= 4\pi(5)^2 \cdot (-2) = -200\pi \ \text{cm}^3/\text{min}$$

Jan 20-11:38 AM

One leg of a right triangle increasing at 4 cm/min.

the other leg is decreasing at 2cm/min.

How fast its area changing when the increasing leg is 6 cm & the decreasing leg is 3cm? $A = \frac{xy}{2}$



$\frac{dx}{dt} = 4$ cm/min. $\frac{dy}{dt} = -2$ cm/min.

$$\frac{dA}{dt} = \frac{d}{dt}\left[\frac{xy}{2}\right]$$

$$= \frac{1}{2}\left[\frac{d}{dt}(xy)\right] = \frac{1}{2}\left[\frac{dx}{dt}\cdot y + x\cdot\frac{dy}{dt}\right]$$

$$= \frac{1}{2}\left[4\cdot 3 + 6\cdot(-2)\right]$$

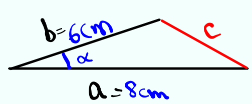
$$= 0 \text{ cm}^2/\text{min.}$$

Jan 20-11:46 AM

Consider a triangle with 6 cm and 8 cm with angle α between them.

Suppose $\frac{d\alpha}{dt} = 1$ Rad/min.

α increases



How fast the third side is changing when $\alpha = 30°$?

Law of Cosine

$$c^2 = a^2 + b^2 - 2ab\cos\alpha$$

$$c^2 = 8^2 + 6^2 - 2\cdot 8\cdot 6\cos\alpha$$

$$c^2 = 100 - 96\cos\alpha$$

$c^2 = 100 - 96\cdot\cos 30°$

$c^2 = 16.862$

$\boxed{c \approx 4}$

$$2c\frac{dc}{dt} = 0 - 96\cdot(-\sin\alpha)\cdot\frac{d\alpha}{dt}$$

$$2\cdot 4\cdot\frac{dc}{dt} = 96\cdot\sin 30°\cdot 1$$

$$2\cdot 4\cdot\frac{dc}{dt} = 48 \qquad 8\frac{dc}{dt} \approx 48$$

$\frac{dc}{dt} \approx 6$ cm/min

Jan 20-11:55 AM

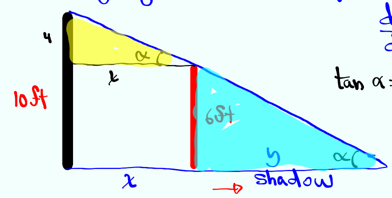
Street light is On.

Street light is 10-ft tall.

A person is 6ft tall and walks away from the light.

How fast is the length of his/her shadow changing if $\underbrace{\text{he/she walks 3 ft/sec?}}$

$\frac{dx}{dt} = 3$



$\tan \alpha = \frac{4}{x} = \frac{6}{y}$

$\frac{4}{x} = \frac{6}{y}$

$4y = 6x$

$\frac{6}{y} = \frac{10}{x+y}$

$10y = 6x + 6y$

$4y = 6x$

$y = \frac{6}{4}x$

$y = \frac{6}{4}x$

$y = \frac{3}{2}x$

$\frac{dy}{dt} = \frac{3}{2} \cdot \frac{dx}{dt}$

$\frac{dy}{dt} = \frac{3}{2} \cdot 3$

$= \frac{9}{2}$ ft/sec.

Jan 20-12:09 PM